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Effective Approach to Constructing n-Mode Wigner Operator in the Entangled State Representation

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Abstract By extending Fan-Klauder entangled state representation to multipartite case. We construct n-mode Wigner operator in the common eigenvector of the multipartite centre-of-mass coordinate and two mass-weighted relative momenta, and its canonical conjugate state, they are both more complicated entangled state of continuum variables. the technique of integration within an ordered product (*IWOP*) of operators is essential in our derivation.

Keywords Wigner operator · Entangled state · IWOP technique

1 Introduction

Wigner function theory [1] has been an important topic since the foundation of quantum mechanics since Wigner function of quantum states (pure states and density operators) are widely used in quantum statistics and quantum optics. Further, Wigner functions and Weyl transforms of operator offer a formulation quantum mechanics in phase space that is equivalent to the standard approach given by the Schrödinger equation [1–5]. The physical meaning of the Wigner distribution function $W(q,p)$ of a particle in pure state $|\psi\rangle\langle\psi|$ lies in that its marginal distributions give the probability of finding the particle with momentum

$$|\psi(p)|^2 = \int_{-\infty}^{\infty} W(q,p)dq, \quad W(q,p) = \langle\psi|\Delta(q,p)|\psi\rangle, \tag{1}$$

and the probability of finding the particle with position

$$|\psi(q)|^2 = \int W(q,p)dp, \tag{2}$$

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respectively. Where $\Delta(q,p)$ is named the Wigner operator. Enlightened by this idea one can immediately wrote down the explicit form of Wigner operator as [6]

$$\Delta(q,p)=\frac{1}{\pi}:e^{-(q-Q)^2-(p-P)^2}:, \tag{3}$$

where the symbol :: denotes normal ordering, $Q=(a+a^\dagger)/\sqrt{2}$, $P=(a-a^\dagger)/(i\sqrt{2})$ are the coordinate and momentum operator, whose eigenvectors are $|q\rangle$ and $|p\rangle$ respectively, $[a,a^\dagger]=1$. One-sided integral of the Wigner operator respectively gives the single-particle's position projection operator

$$\int_{-\infty}^{\infty}dp\Delta(q,p)=\frac{1}{\sqrt{\pi}}:e^{-(q-Q)^2}:=|q\rangle\langle q|, \tag{4}$$

and the momentum projection operator

$$\int_{-\infty}^{\infty}dq\Delta(q,p)=\frac{1}{\sqrt{\pi}}:e^{-(p-P)^2}:=|p\rangle\langle p|. \tag{5}$$

Recently the concept of quantum entanglement has aroused much attention of physicists since it is useful in quantum communication and quantum information. The bipartite entangled state representation is constructed in Ref. [7–12], which was enlightened by Einstein-Podolsky-Rosen's argument that two particles' relative coordinate Q_1-Q_2 and the total momentum P_1+P_2 are commutable and can be simultaneously measured [13]. Generally speaking, as pointed in Ref. [14, 15] that for an entangled particles system, the Wigner operator should be put into the corresponding entangled state representation such that the marginal distributions of the Wigner function would give the correct probability in the sense of finding entangled particles simultaneously. Thus introducing entangled state representation is inevitable. The aim of this paper is to construct phase space formalism of Wigner operator in multipartite entangled system, which is different from that in Ref. [10].

In the following we shall show that with the help of the technique of integration within an ordered product (*IWOP*) of operators [16, 17], the normally ordered Gaussian form of quantum mechanical completeness relation provides us with much convenience to set up multipartite Wigner operator and the corresponding phase space formalism.

To illustrate our approach clearly, as an example, we in Sect. 2 demonstrate how the Wigner operator for bipartite entangled system can be constructed from two mutually conjugate entangled state representations whose completeness relation exhibiting complex Gaussian form within normal ordering. In Sects. 3–4 we discuss tripartite case and multipartite case respectively.

2 Approach for Deriving Wigner Operator for Bipartite Entangled System

In Ref. [7–12] Fan and Klauder have constructed the bipartite entangled state

$$|\eta\rangle=\exp[-|\eta|^2/2+\eta a_1^\dagger-\eta^* a_2^\dagger+a_1^\dagger a_2^\dagger]|00\rangle, \tag{6}$$

such a state may be figured out by thinking of the following eigenvector equations

$$(a_1-a_2^\dagger-\eta)|\eta\rangle=0, \qquad (a_1^\dagger-a_2-\eta^*)|\eta\rangle=0. \tag{7}$$

and then to make up a Gaussian integration with normal ordering which is equal to unit, i.e.,

$$\int \frac{d^2\eta}{\pi} : \exp\{-[\eta - (a_1 - a_2^\dagger)][\eta^* - (a_1^\dagger - a_2)]\} := 1, \tag{8}$$

using $: e^{-a_1^\dagger a_1 - a_2^\dagger a_2} :$ to decomposing the exponential in (8) we see the completeness relation $\int \frac{d^2\eta}{\pi}|\eta\rangle\langle\eta| = 1$. Since $|\eta\rangle$ is also the common eigenvector of $(Q_1 - Q_2)$ and $(P_1 + P_2)$,

$$(Q_1 - Q_2)|\eta\rangle = \sqrt{2}\eta_1|\eta\rangle, \qquad (P_1 + P_2)|\eta\rangle = \sqrt{2}\eta_2|\eta\rangle, \quad \eta = \eta_1 + i\eta_2.$$

Equation (8) can also be recast into

$$\int \frac{d^2\eta}{\pi}|\eta\rangle\langle\eta| = \iint_{-\infty}^{\infty} \frac{d\eta_1 d\eta_2}{\pi} : e^{-[(\eta_1 - \frac{Q_1 - Q_2}{\sqrt{2}})^2 + (\eta_2 - \frac{P_1 + P_2}{\sqrt{2}})^2]} := 1, \quad d^2\eta = d\eta_1 d\eta_2. \tag{9}$$

On the other hand, if we merely change the sign before a_2 and $a_2^\dagger$ in (8), the resolvent of unit still keeps, i.e.,

$$\int \frac{d^2\xi}{\pi} : \exp\{-[\xi - (a_1 + a_2^\dagger)][\xi^* - (a_1^\dagger + a_2)]\} := 1, \tag{10}$$

which leads to the conjugate state of $|\eta\rangle$ i.e.,

$$|\xi\rangle = \exp\left[-\frac{|\xi|^2}{2} + \xi a_1^\dagger + \xi^* a_2^\dagger - a_1^\dagger a_2^\dagger\right]|0\rangle, \quad \xi = \xi_1 + i\xi_2. \tag{11}$$

$|\xi\rangle$ is also an entangled state representation and it obeys the eigen-equations

$$(a_1 + a_2^\dagger)|\xi\rangle = \xi|\xi\rangle, \qquad (a_1^\dagger + a_2)|\xi\rangle = \xi^*|\xi\rangle, \tag{12}$$

or

$$(Q_1 + Q_2)|\xi\rangle = \sqrt{2}\xi_1|\xi\rangle, \qquad (P_1 - P_2)|\xi\rangle = \sqrt{2}\xi_2|\xi\rangle, \tag{13}$$

so we also have

$$\iint_{-\infty}^{\infty} \frac{d\xi_1 d\xi_2}{\pi} : e^{-[(\xi_1 - \frac{Q_1 + Q_2}{\sqrt{2}})^2 + (\xi_2 - \frac{P_1 - P_2}{\sqrt{2}})^2]} := 1. \tag{14}$$

Combining the right hand sides of (9) and (14) together as we did in (3) we introduce the Gaussian form of normally ordered operator

$$\frac{1}{\pi^2} : \exp\left[-\left(\sigma_1 - \frac{Q_1 - Q_2}{\sqrt{2}}\right)^2 - \left(\sigma_2 - \frac{P_1 + P_2}{\sqrt{2}}\right)^2 - \left(\gamma_1 - \frac{Q_1 + Q_2}{\sqrt{2}}\right)^2 - \left(\gamma_2 - \frac{P_1 - P_2}{\sqrt{2}}\right)^2\right] :\equiv \Delta_{12}(\sigma, \gamma). \tag{15}$$

It is easy to see that (15) is equal to

$$\Delta_{12}(\sigma, \gamma) = \frac{1}{\pi^2} : \exp[-(\sigma - a_1 + a_2^\dagger)(\sigma^* - a_1^\dagger + a_2) - (\gamma - a_1 - a_2^\dagger)(\gamma^* - a_1^\dagger - a_2)] :, \tag{16}$$

where $\sigma = \sigma_1 + i\sigma_2$, $\gamma = \gamma_1 + i\gamma_2$. We name $\Delta_{12}(\sigma,\gamma)$ the entangled Wigner operator, since its marginal distributions are

$$\int d^2\sigma\,\Delta_{12}(\sigma,\gamma) = \frac{1}{\pi} : \exp[-(\gamma - a_2^\dagger - a_1)(\gamma^* - a_1^\dagger - a_2)] := \frac{1}{\pi}|\xi\rangle\langle\xi|_{\xi=\gamma}, \tag{17}$$

and

$$\int d^2\gamma\,\Delta_{12}(\sigma,\gamma) = \frac{1}{\pi} : \exp[-(\sigma + a_2^\dagger - a_1)(\sigma^* - a_1^\dagger + a_2)] := \frac{1}{\pi}|\eta\rangle\langle\eta|_{\eta=\sigma}. \tag{18}$$

The corresponding marginal distributions of Wigner function of bipartite state $|\psi\rangle$ are

$$\int d^2\sigma\,\langle\psi|\Delta_{12}(\sigma,\gamma)|\psi\rangle = \frac{1}{\pi}|\psi(\xi)|^2_{\xi=\gamma}, \tag{19}$$

and

$$\int d^2\gamma\,\langle\psi|\Delta_{12}(\sigma,\gamma)|\psi\rangle = \frac{1}{\pi}|\psi(\eta)|^2_{\eta=\sigma}, \tag{20}$$

where $|\psi(\xi)|^2$ denotes the measurement probability of two particles' center-of-mass coordinate $\sqrt{2}\xi_1$ and relative momentum $\sqrt{2}\xi_2$, while $|\psi(\eta)|^2$ denotes the measurement probability of total momentum $\sqrt{2}\eta_2$ and simultaneously relative position $\sqrt{2}\eta_1$. We emphasize that, for an entangled-particle system, the physical meaning of the Wigner distribution function should lie in that its marginal distributions would give the probability of finding the particles in an entangled way.

Note that $\Delta_{12}(\sigma,\gamma)$ is equal to the product of two one-mode Wigner operators, $\Delta_{12}(\sigma,\gamma) = \Delta_1(\alpha,\alpha^*)\Delta_2(\beta,\beta^*)$ when we set $\sigma \equiv \alpha + \beta^*$, $\gamma \equiv \alpha - \beta^*$, $\alpha \equiv \frac{1}{\sqrt{2}}(q_1 + ip_1)$, $\beta \equiv \frac{1}{\sqrt{2}}(q_2 + ip_2)$.

The above description presented a concise approach for introducing Wigner operators for bipartite entangled system. The procedure can be generalized as: (1) Writing explicitly the common-eigenvector equations for associated two-particle observables which are commutative; (2) According to these equations writing explicitly the normally ordered Gaussian form of completeness relations of two conjugate entangled state representations; (3) Combining these two Gaussian forms to compose the entangled Wigner operator.

3 The Wigner Operator for the Mass-Dependent Tripartite Entangled System Derived by Virtue of the IWOP Technique

Enlightened by the method of finding the Wigner operator for bipartite entangled system we now search for Wigner operator for the mass-dependent tripartite entangled system. Let $|\chi, p_2, p_3\rangle$ be the common eigenvector of the three compatible operators: $(\frac{P_1}{\mu_1} - \frac{P_2}{\mu_2})$, $(\frac{P_1}{\mu_1} - \frac{P_3}{\mu_3})$ and $(\mu_1 X_1 + \mu_2 X_2 + \mu_3 X_3)$, where $\mu_i = m_i/M$ $(i = 1, 2, 3)$, is the reduced mass of each particle, $M = \sum_{i=1}^3 m_i$ is the total mass of 3 particles, $\mu_1 X_1 + \mu_2 X_2 + \mu_3 X_3$ is the tripartite centre-of-mass coordinate and $(\frac{P_1}{\mu_1} - \frac{P_2}{\mu_2})$, $(\frac{P_1}{\mu_1} - \frac{P_3}{\mu_3})$ are the mass-weighted relative momenta. The state $|\chi, p_2, p_3\rangle$ obeys the eigenvector equations:

$$\left(\frac{P_1}{\mu_1} - \frac{P_2}{\mu_2}\right)|\chi, p_2, p_3\rangle = p_2|\chi, p_2, p_3\rangle,$$

$$\left(\frac{P_1}{\mu_1}-\frac{P_3}{\mu_3}\right)|\chi,p_2,p_3\rangle = p_3|\chi,p_2,p_3\rangle,$$
$$(\mu_1X_1+\mu_2X_2+\mu_3X_3)\,|\chi,p_2,p_3\rangle = \chi|\chi,p_2,p_3\rangle. \tag{21}$$

Note that (21) also implies another non-independent equation

$$\left(\frac{P_2}{\mu_2}-\frac{P_3}{\mu_3}\right)|\chi,p_2,p_3\rangle = (p_3-p_2)\,|\chi,p_2,p_3\rangle. \tag{22}$$

We can rewrite (21) and (22) in the following form

$$\begin{aligned}
(\mu_2P_1-\mu_1P_2)\,|\chi,p_2,p_3\rangle &= \mu_1\mu_2p_2|\chi,p_2,p_3\rangle,\\
(\mu_3P_1-\mu_1P_3)\,|\chi,p_2,p_3\rangle &= \mu_1\mu_3p_3|\chi,p_2,p_3\rangle,\\
(\mu_1X_1+\mu_2X_2+\mu_3X_3)\,|\chi,p_2,p_3\rangle &= \chi|\chi,p_2,p_3\rangle,\\
(\mu_3P_2-\mu_2P_3)\,|\chi,p_2,p_3\rangle &= \mu_2\mu_3\,(p_3-p_2)\,|\chi,p_2,p_3\rangle.
\end{aligned} \tag{23}$$

Using (21) and (22) and the *IWOP* technique we can construct the unit integration as

$$\begin{aligned}
\iiint_{-\infty}^{\infty}\frac{d\chi dp_2dp_3}{\pi^{\frac{3}{2}}}\frac{\mu_1\mu_2\mu_3}{\lambda}:\exp\Big\{&-\frac{1}{\lambda}[\mu_1\mu_2p_2-(\mu_2P_1-\mu_1P_2)]^2\\
&-\frac{1}{\lambda}[\mu_1\mu_3p_3-(\mu_3P_1-\mu_1P_3)]^2-\frac{1}{\lambda}[\chi-(\mu_1X_1+\mu_2X_2+\mu_3X_3)]^2\\
&-\frac{1}{\lambda}[\mu_2\mu_3(p_3-p_2)-(\mu_3P_2-\mu_2P_3)]^2\Big\}:=1,
\end{aligned} \tag{24}$$

where $\lambda=\sum_{i=1}^{3}\mu_i^2$. The correctness of (24) can be directly confirmed by performing the Gaussian integration within :: in it. Then we decompose the integrand in (24) as the form "function of creation operators $\times:\exp(-\sum_{i=1}^{3}a_i^{\dagger}a_i):\times$ function of annihilation operators".

The integrand in (24)

$$\begin{aligned}
&=\exp\Big\{A+\frac{\sqrt{2}\chi}{\lambda}\sum_{i=1}^{3}\mu_ia_i^{\dagger}+\frac{i\sqrt{2}p_2\mu_2}{\lambda}[\mu_1\mu_2a_1^{\dagger}-(\mu_1^2+\mu_3^2)a_2^{\dagger}+\mu_3\mu_2a_3^{\dagger}]\\
&\quad+\frac{i\sqrt{2}p_3\mu_3}{\lambda}[\mu_1\mu_3a_1^{\dagger}+\mu_2\mu_3a_2^{\dagger}-(\mu_1^2+\mu_2^2)a_3^{\dagger}]+S^{\dagger}\Big\}\\
&\quad\times:\exp\Big\{-\sum_{i=1}^{3}a_i^{\dagger}a_i\Big\}:\\
&\quad\times\exp\Big\{A+\frac{\sqrt{2}\chi}{\lambda}\sum_{i=1}^{3}\mu_ia_i-\frac{i\sqrt{2}p_2\mu_2}{\lambda}[\mu_1\mu_2a_1-(\mu_1^2+\mu_3^2)a_2+\mu_3\mu_2a_3]\\
&\quad-\frac{i\sqrt{2}p_3\mu_3}{\lambda}[\mu_1\mu_3a_1+\mu_2\mu_3a_2-(\mu_1^2+\mu_2^2)a_3]+S\Big\}\equiv|\chi,p_2,p_3\rangle\langle\chi,p_2,p_3|,
\end{aligned} \tag{25}$$

where

$$
\begin{aligned}
-\frac{\chi^2}{2\lambda}-\frac{1}{2\lambda}[(\mu_1^2+\mu_3^2)\mu_2^2p_2^2+(\mu_1^2+\mu_2^2)\mu_3^2p_3^2-2\mu_2^2\mu_3^2p_2p_3]\equiv A,\\
-\frac{2}{\lambda}\sum_{i<j}^{3}\mu_i\mu_j a_i a_j+\sum_{i=1}^{3}\left(\frac{1}{2}-\frac{\mu_i^2}{\lambda}\right)a_i^2\equiv S,
\end{aligned}
\tag{26}
$$

and

$$
\begin{aligned}
|\chi,p_2,p_3\rangle &= \pi^{-\frac{3}{4}}\sqrt{\frac{\mu_1\mu_2\mu_3}{\lambda}}\exp\left[A+\frac{\sqrt{2}\chi}{\lambda}\sum_{i=1}^{3}\mu_i a_i^{\dagger}\right.\\
&\quad+\frac{i\sqrt{2}p_2\mu_2}{\lambda}[\mu_1\mu_2a_1^{\dagger}-(\mu_1^2+\mu_3^2)a_2^{\dagger}+\mu_3\mu_2a_3^{\dagger}]\\
&\quad\left.+\frac{i\sqrt{2}p_3\mu_3}{\lambda}[\mu_1\mu_3a_1^{\dagger}+\mu_2\mu_3a_2^{\dagger}-(\mu_1^2+\mu_2^2)a_3^{\dagger}]+S^{\dagger}\right]|000\rangle,
\end{aligned}
\tag{27}
$$

here we have used $:\exp(-\sum_{i=1}a_i^{\dagger}a_i):=|000\rangle\langle000|$. The form of $|\chi,p_2,p_3\rangle$ in (27) is the same as that of Ref. [10].

By observing that the operators

$$
\left(\frac{X_1}{\mu_1}-\frac{X_2}{\mu_2}\right),\quad\left(\frac{X_1}{\mu_1}-\frac{X_3}{\mu_3}\right)\quad\text{and}\quad(\mu_1P_1+\mu_2P_2+\mu_3P_3)
$$

are also permutable with each other, in similar to (24) we have another form of unit integration

$$
\begin{aligned}
&\iiint_{-\infty}^{\infty}\frac{dp\,d\chi_2d\chi_3}{\pi^{\frac{3}{2}}}\frac{\mu_1\mu_2\mu_3}{\lambda}:\exp\left\{-\frac{1}{\lambda}[\mu_1\mu_2\chi_2-(\mu_2X_1-\mu_1X_2)]^2\right.\\
&\quad-\frac{1}{\lambda}[\mu_1\mu_3\chi_3-(\mu_3X_1-\mu_1X_3)]^2-\frac{1}{\lambda}[p-(\mu_1P_1+\mu_2P_2+\mu_3P_3)]^2\\
&\quad\left.-\frac{1}{\lambda}[\mu_2\mu_3(\chi_3-\chi_2)-(\mu_3X_2-\mu_2X_3)]^2\right\}:=1,
\end{aligned}
\tag{28}
$$

which leads to the appearance

$$
\begin{aligned}
|p,\chi_2,\chi_3\rangle = \pi^{-\frac{3}{4}}\sqrt{\tfrac{\mu_1\mu_2\mu_3}{\lambda}}\exp\left[A+\tfrac{\sqrt{2}p}{\lambda}\textstyle\sum_{i=1}^{3}\mu_ia_i^{\dagger}+\tfrac{i\sqrt{2}\chi_2\mu_2}{\lambda}[\mu_1\mu_2a_1^{\dagger}-(\mu_1^2+\mu_3^2)a_2^{\dagger}\right.\\
\left.+\mu_3\mu_2a_3^{\dagger}]+\tfrac{i\sqrt{2}\chi_3\mu_3}{\lambda}[\mu_1\mu_3a_1^{\dagger}+\mu_2\mu_3a_2^{\dagger}-(\mu_1^2+\mu_2^2)a_3^{\dagger}]+S^{\dagger}\right]|000\rangle,
\end{aligned}
\tag{29}
$$

$|p,\chi_2,\chi_3\rangle$ is the conjugate state of $|\chi,p_2,p_3\rangle$. Similarly, combining the right hand sides of (24) and (28) together as before, we now introduce the Gaussian form of normally ordered operator

$$\begin{aligned}
&\Delta_3(p,\chi_2,\chi_3;\chi,p_2,p_3)\\
&\quad=\frac{1}{\pi^3}\left(\frac{\prod_{i=1}^3\mu_i}{\lambda}\right)^2:\exp\left\{-\frac{1}{\lambda}\sum_{i=2}^3[\mu_1\mu_i p_i-(\mu_i P_1-\mu_1 P_i)]^2\right.\\
&\qquad-\frac{1}{\lambda}\left(\chi-\sum_{i=1}^3\mu_i X_i\right)^2-\frac{1}{\lambda}\sum_{i=2,j>i}^3[\mu_i\mu_j(p_j-p_i)-(\mu_j P_i-\mu_i P_j)]^2\\
&\qquad-\frac{1}{\lambda}\sum_{i=2}^3[\mu_1\mu_i\chi_i-(\mu_i X_1-\mu_1 X_i)]^2-\frac{1}{\lambda}\left(\chi-\sum_{i=1}^3\mu_i P_i\right)^2\\
&\qquad\left.-\frac{1}{\lambda}\sum_{i=2,j>i}^3[\mu_i\mu_j(\chi_j-\chi_i)-(\mu_j X_i-\mu_i X_j)]^2\right\}:,
\end{aligned}\tag{30}$$

which is the mass-dependent tripartite entangled Wigner operator, since its marginal distributions are

$$\iiint\frac{d\chi}{\pi^{\frac{3}{2}}}\frac{\mu_1\mu_2\mu_3}{\lambda}\prod_{i=2}^3 dp_i\,\Delta_3(p,\chi_2,\chi_3;\chi,p_2,p_3)=|p,\chi_2,\chi_3\rangle\langle p,\chi_2,\chi_3|,\tag{31}$$

and

$$\iiint\frac{dp}{\pi^{\frac{3}{2}}}\frac{\mu_1\mu_2\mu_3}{\lambda}\prod_{i=2}^3 d\chi_i\,\Delta_3(p,\chi_2,\chi_3;\chi,p_2,p_3)=|\chi,p_2,p_3\rangle\langle\chi,p_2,p_3|\tag{32}$$

we can further confirm $\Delta_3=\Delta_1(\alpha_1)\Delta_2(\alpha_2)\Delta_3(\alpha_3)$ when we set $\alpha_j=\frac{q_j+ip_j}{\sqrt{2}}$.

4 The Wigner Operator for Multipartite Mass-Dependent Entangled System Derived from Gaussian Form of Completeness Relation

Hinted by the way of finding tripartite entangled states through their Gaussian-form completeness relation in normal ordering, we now search for the common eigenvector $|\chi,p_2,p_3,\ldots,p_n\rangle$ of the n compatible operators: $(\frac{P_1}{\mu_1}-\frac{P_2}{\mu_2})$, $(\frac{P_1}{\mu_1}-\frac{P_3}{\mu_3})$, $\ldots$, $(\frac{P_1}{\mu_1}-\frac{P_i}{\mu_i})$ and $\sum_{i=1}^n\mu_i X_i$, where $\mu_i=m_i/M$ $(i=1,2,3,\ldots,n)$, is each particle's reduced mass, $M=\sum_{i=1}^n m_i$ is the total mass of n particles, $\sum_{i=1}^n\mu_i X_i$ is the n-partite's centre-of-mass coordinate and $(\frac{P_1}{\mu_1}-\frac{P_2}{\mu_2})$, $(\frac{P_1}{\mu_1}-\frac{P_3}{\mu_3}),\ldots,(\frac{P_1}{\mu_1}-\frac{P_i}{\mu_i})$ are the mass-weighted relative momenta. The $|\chi,p_2,p_3,\ldots,p_n\rangle$ obeys the following eigenvector equations

$$\begin{aligned}
\left(\frac{P_1}{\mu_1}-\frac{P_2}{\mu_2}\right)|\chi,p_2,p_3,\ldots,p_n\rangle&=p_2|\chi,p_2,p_3,\ldots,p_n\rangle,\\
\left(\frac{P_1}{\mu_1}-\frac{P_3}{\mu_3}\right)|\chi,p_2,p_3,\ldots,p_n\rangle&=p_3|\chi,p_2,p_3,\ldots,p_n\rangle,\\
&\vdots
\end{aligned}$$

$$\left(\frac{P_1}{\mu_1}-\frac{P_n}{\mu_n}\right)|\chi,p_2,p_3,\dots,p_n\rangle = p_n|\chi,p_2,p_3,\dots,p_n\rangle,$$
$$\sum_{i=1}^{n}\mu_i X_i|\chi,p_2,p_3,\dots,p_n\rangle=\chi|\chi,p_2,p_3,\dots,p_n\rangle. \tag{33}$$

Equation (33) implies another non-independent equation

$$\left(\frac{P_i}{\mu_i}-\frac{P_j}{\mu_j}\right)|\chi,p_2,p_3,\dots,p_n\rangle=(p_j-p_i)|\chi,p_2,p_3,\dots,p_n\rangle, \tag{34}$$

where $j>i$ and i starts from 2 to n. Here we can also rewrite (33) and (34) in the form

$$\begin{aligned}
(\mu_2P_1-\mu_1P_2)|\chi,p_2,p_3,\dots,p_n\rangle &= \mu_1\mu_2p_2|\chi,p_2,p_3,\dots,p_n\rangle,\\
(\mu_3P_1-\mu_1P_3)|\chi,p_2,p_3,\dots,p_n\rangle &= \mu_1\mu_3p_3|\chi,p_2,p_3,\dots,p_n\rangle,\\
&\vdots\\
(\mu_nP_1-\mu_1P_n)|\chi,p_2,p_3,\dots,p_n\rangle &= \mu_1\mu_np_n|\chi,p_2,p_3,\dots,p_n\rangle,\\
\sum_{i=1}^{n}\mu_iX_i|\chi,p_2,p_3,\dots,p_n\rangle &= \chi|\chi,p_2,p_3,\dots,p_n\rangle,\\
(\mu_jP_i-\mu_iP_j)|\chi,p_2,p_3,\dots,p_n\rangle &= \mu_i\mu_j(p_j-p_i)|\chi,p_2,p_3,\dots,p_n\rangle.
\end{aligned} \tag{35}$$

Generalizing the form of (35) we write down the following normally ordered form of n-fold integration by exhausting all the eigenvalue equations

$$\begin{aligned}
&\int\cdots\int_{-\infty}^{\infty}\frac{d\chi}{\pi^{\frac{n}{2}}}\prod_{i=2}^{n}dp_i\frac{\prod_{i=1}^{n}\mu_i}{\lambda}:\exp\left\{-\frac{1}{\lambda}\sum_{i=2}^{n}[\mu_1\mu_ip_i-(\mu_iP_1-\mu_1P_i)]^2\right.\\
&\left.-\frac{1}{\lambda}\left(\chi-\sum_{i=1}^{n}\mu_iX_i\right)^2-\frac{1}{\lambda}\sum_{i=2,j>i}^{n}[\mu_i\mu_j(p_j-p_i)-(\mu_jP_i-\mu_iP_j)]^2\right\}:=1,
\end{aligned} \tag{36}$$

where $\lambda=\sum_{i=1}^{n}\mu_i^2$, its correctness (equal to unity) has been confirmed in Ref. [18].

Now using

$$:\exp\left(-\sum_{i=1}^{n}a_i^{\dagger}a_i\right):=|00\cdots0\rangle\langle00\cdots0|$$

we decompose the integrand in (36) as the form "function of creation operators $\times:\exp(-\sum_{i=1}^{n}a_i^{\dagger}a_i):\times$ function of annihilation operators" which can re-express (36) as

$$1=\int\cdots\int_{-\infty}^{\infty}dp\prod_{i=2}^{n}d\chi_i|p,\chi_2,\chi_3,\dots,\chi_n\rangle\langle p,\chi_2,\chi_3,\dots,\chi_n|, \tag{37}$$

where

$$|\chi, p_2, p_3, \ldots, p_n\rangle = \pi^{-\frac{n}{4}}\sqrt{\frac{\prod_{i=1}^n \mu_i}{\lambda}} \exp\left[M + \frac{\sqrt{2}\chi}{\lambda}\sum_{i=1}^n \mu_i a_i^\dagger \right.$$
$$+ \frac{i\sqrt{2}}{\lambda}\sum_{j=2}^n \mu_j^2 p_j\left(\sum_{i=1}^n \mu_i a_i^\dagger - \frac{\lambda}{\mu_j}a_j^\dagger\right)$$
$$\left. - \frac{2}{\lambda}\sum_{i<j,i,j=1}^n \mu_i\mu_j a_i^\dagger a_j^\dagger + \sum_{i=1}^n\left(\frac{1}{2} - \frac{\mu_i^2}{\lambda}\right)a_i^{\dagger 2}\right]|00\cdots 0\rangle, \quad (38)$$

here

$$-\frac{\chi^2}{2\lambda} - \frac{1}{2\lambda}\sum_{i=2}^n(\lambda - \mu_i^2)\mu_i^2 p_i^2 + \frac{1}{\lambda}\sum_{i<j,i,j=2}^n \mu_i^2\mu_j^2 p_i p_j \equiv M,$$
$$-\frac{2}{\lambda}\sum_{i<j,i,j=1}^n \mu_i\mu_j a_i a_j + \sum_{i=1}^n\left(\frac{1}{2} - \frac{\mu_i^2}{\lambda}\right)a_i^2 \equiv N. \quad (39)$$

Similarly, by observing that the operators

$$\left(\frac{X_1}{\mu_1} - \frac{X_2}{\mu_2}\right), \quad \left(\frac{X_1}{\mu_1} - \frac{X_3}{\mu_3}\right), \quad \ldots, \quad \left(\frac{X_1}{\mu_1} - \frac{X_i}{\mu_i}\right)$$

and

$$\sum_{i=1}^n \mu_i P_i,$$

are also permutable with each other, we can construct the following eigenvector equations

$$(\mu_2 X_1 - \mu_1 X_2)|p, \chi_2, \chi_3, \ldots, \chi_n\rangle = \mu_1\mu_2\chi_2|p, \chi_2, \chi_3, \ldots, \chi_n\rangle,$$
$$(\mu_3 X_1 - \mu_1 X_3)|p, \chi_2, \chi_3, \ldots, \chi_n\rangle = \mu_1\mu_3\chi_3|p, \chi_2, \chi_3, \ldots, \chi_n\rangle,$$
$$\vdots$$
$$(\mu_n X_1 - \mu_1 X_n)|p, \chi_2, \chi_3, \ldots, \chi_n\rangle = \mu_1\mu_n\chi_n|p, \chi_2, \chi_3, \ldots, \chi_n\rangle, \quad (40)$$
$$\sum_{i=1}^n \mu_i P_i|p, \chi_2, \chi_3, \ldots, \chi_n\rangle = p|p, \chi_2, \chi_3, \ldots, \chi_n\rangle,$$
$$(\mu_j X_i - \mu_i X_j)|p, \chi_2, \chi_3, \ldots, \chi_n\rangle = \mu_i\mu_j(\chi_j - \chi_i)|p, \chi_2, \chi_3, \ldots, \chi_n\rangle.$$

In similar to (36) we have the unity

$$\int\cdots\int_{-\infty}^{\infty}\frac{dp}{\pi^{\frac{n}{2}}}\prod_{i=2}^n d\chi_i\frac{\prod_{i=1}^n\mu_i}{\lambda} : \exp\left\{-\frac{1}{\lambda}\sum_{i=2}^n[\mu_1\mu_i\chi_i - (\mu_i X_1 - \mu_1 X_i)]^2\right.$$
$$\left. - \frac{1}{\lambda}\left(p - \sum_{i=1}^n \mu_i P_i\right)^2 - \frac{1}{\lambda}\sum_{i=2, j>i}^n[\mu_i\mu_j(\chi_j - \chi_i) - (\mu_j X_i - \mu_i X_j)]^2\right\} := 1, \quad (41)$$

which leads to the n-partite entangled state

$$|p,\chi_2,\chi_3,\dots,\chi_n\rangle=\pi^{-\frac{n}{4}}\sqrt{\frac{\prod_{i=1}^n\mu_i}{\lambda}}\exp\left[M+\frac{\sqrt{2}p}{\lambda}\sum_{i=1}^n\mu_i a_i^\dagger\right.$$
$$+\frac{i\sqrt{2}}{\lambda}\sum_{j=2}^n\mu_j^2\chi_j\left(\sum_{i=1}^n\mu_i a_i^\dagger-\frac{\lambda}{\mu_j}a_j^\dagger\right)$$
$$\left.-\frac{2}{\lambda}\sum_{i<j,i,j=1}^n\mu_i\mu_j a_i^\dagger a_j^\dagger+\sum_{i=1}^n\left(\frac{1}{2}-\frac{\mu_i^2}{\lambda}\right)a_i^{\dagger 2}\right]|00\cdots 0\rangle. \tag{42}$$

$|\chi,p_2,p_3,\dots,p_n\rangle$ is the canonical conjugate state of $|p,\chi_2,\chi_3,\dots,\chi_n\rangle$. The Wigner operator is then composed of the right hand side of (36) and that of (41), i.e.

$$\Delta_n(p,\chi_2,\chi_3,\dots,\chi_n;\chi,p_2,p_3,\dots,p_n)$$
$$=\frac{1}{\pi^n}\left(\frac{\prod_{i=1}^n\mu_i}{\lambda}\right)^2:\exp\left\{-\frac{1}{\lambda}\sum_{i=2}^n[\mu_1\mu_i p_i-(\mu_i P_1-\mu_1 P_i)]^2\right.$$
$$-\frac{1}{\lambda}\left(\chi-\sum_{i=1}^n\mu_i X_i\right)^2-\frac{1}{\lambda}\sum_{i=2,j>i}^n[\mu_i\mu_j(p_j-p_i)-(\mu_j P_i-\mu_i P_j)]^2$$
$$-\frac{1}{\lambda}\sum_{i=2}^n[\mu_1\mu_i\chi_i-(\mu_i X_1-\mu_1 X_i)]^2-\frac{1}{\lambda}\left(\chi-\sum_{i=1}^n\mu_i P_i\right)^2$$
$$\left.-\frac{1}{\lambda}\sum_{i=2,j>i}^n[\mu_i\mu_j(\chi_j-\chi_i)-(\mu_j X_i-\mu_i X_j)]^2\right\}:. \tag{43}$$

The marginal distributions of $\Delta_n(p,\chi_2,\chi_3,\dots,\chi_n;\chi,p_2,p_3,\dots,p_n)$ are

$$\int\cdots\int_{-\infty}^{\infty}\frac{d\chi}{\pi^{\frac{n}{2}}}\frac{\prod_{i=1}^n\mu_i}{\lambda}\prod_{i=2}^n dp_i\,\Delta_n(p,\chi_2,\chi_3,\dots,\chi_n;\chi,p_2,p_3,\dots,p_n)$$
$$=|p,\chi_2,\chi_3,\dots,\chi_n\rangle\langle p,\chi_2,\chi_3,\dots,\chi_n|, \tag{44}$$

and

$$\int\cdots\int_{-\infty}^{\infty}\frac{dp}{\pi^{\frac{n}{2}}}\frac{\prod_{i=1}^n\mu_i}{\lambda}\prod_{i=2}^n d\chi_i\,\Delta_n(p,\chi_2,\chi_3,\dots,\chi_n;\chi,p_2,p_3,\dots,p_n)$$
$$=|\chi,p_2,p_3,\dots,p_n\rangle\langle\chi,p_2,p_3,\dots,p_n|. \tag{45}$$

The corresponding marginal distributions of Wigner function of n-partite state $|\psi\rangle$ are $|\langle\chi,p_2,p_3,\dots,p_n|\Psi\rangle|^2|_{\chi=q_0,p_j=p_j-p_1}$ and $|\langle p,\chi_2,\chi_3,\dots,\chi_n|\Psi\rangle|^2|_{p=p_0,\chi_j=q_j-q_1}$, which is proportional to the probability of finding n-partite system with center-of-mass position q_0 [relative position χ_j $(j=2,3,\dots,n)$] and simultaneously relative momentums p_j $(j=2,3,\dots,n)$ [total momentums p_0].

In summary, with the help of the *IWOP* technique we have presented a concise approach to introducing Wigner operators with n mass-different partites' entangled representation.

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